Homework Assignments

Bifurcations: Theory and Applications

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`http://dynamics.mi.fu-berlin.de/lectures/`

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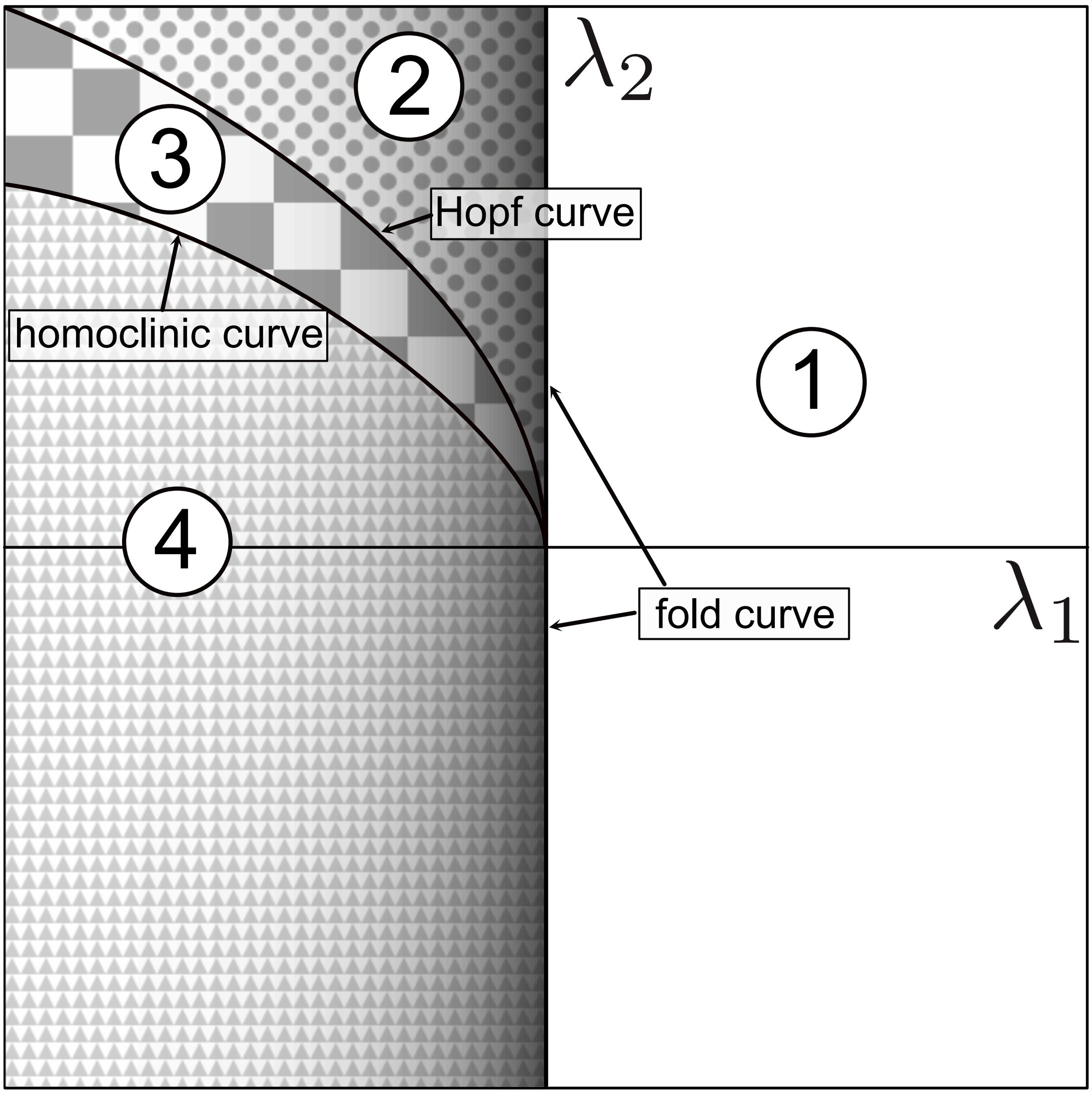
In problems 17–19 consider the unfolding of the normal form for the Arnol'd-Takens-Bogdanov bifurcation

$$\dot{x}_1 = x_2,$$
$$\dot{x}_2 = \lambda_1 + \lambda_2 x_2 + x_1^2 + x_1 x_2.$$

Assume $\lambda_1, \lambda_2, x_1, x_2$ are near 0. In the lecture we discussed the splitting of the parameter plane $\lambda = (\lambda_1, \lambda_2)$ into four regions in which the system showcases fundamentally different dynamics.



Problem 17: Fix $\lambda_2 \neq 0$ and consider $-\delta(\lambda_2) < \lambda_1 < 0$, for small enough $\delta = \delta(\lambda_2)$. Use center manifold theory to prove that there exists a heteroclinic orbit connecting the two equilibria x_+ and x_-. Let ① denote the region $\lambda_1 > 0$ in the parameter plane. Sketch the phase portraits in a neighborhood of the origin as the parameters (λ_1, λ_2) cross from region ① to ② and from ① to ④, respectively.

Hint: Use the invariance of the center manifold to obtain the coefficients of its Taylor expansion of the form

$$\eta(\lambda_1, x_1) = h_{01}\lambda_1 + h_{20}x_1^2 + h_{11}x_1\lambda_1 + h_{02}\lambda_1^2 + \ldots,$$

with coefficients h_{jk} which depend on λ_2.

Problem 18: Sketch the phase portraits in a neighborhood of the origin as the parameters cross from region ② to ③, by crossing the Hopf curve.

Problem 19: Try to guess the phase portraits in a neighborhood of the origin as the parameters cross from region ③ to ④, by crossing the homoclinic curve.

Problem 20: Consider the real ordinary differential equation

$$\dot{x} = \lambda_1 + \lambda_2 x + x^2.$$

Determine the number of equilibria and their stability, depending on the values of the real parameters λ_1 and λ_2. Sketch the phase portraits close to the origin for each relevant region in the parameter plane, in the spirit of problems 17–19.

References:
J. Guckenheimer and P. Holmes: Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields, Springer, 2003, Sec. 7.3.
V.I. Arnol'd: Geometrical Methods in the Theory of Ordinary Differential Equations, Springer, 1988.